Problem set 3

December 1, 2015

Problem 1. Let G be a group. Show that if G is non-abelian, then the maps defined by $g.a = ag$ for all $g, a \in G$ do not satisfy the axioms of a (left) group action of G on itself.

Problem 2. Let S_n act on $\mathbb{R}^n$ by permuting coordinates of vectors: for $\sigma \in S_n$ and $v = (c_1, \ldots, c_n)$ let $\sigma.v$ to be $\sigma.v = (c_{\sigma(1)}, \ldots, c_{\sigma(n)})$. Check that this is **not** an action of S_n on $\mathbb{R}^n$ according to our definition! (Hint: take $n = 3$, $\sigma_1 = (12)$ and $\sigma_2 = (23)$ and see that something goes wrong.)

What goes wrong in general?

Prove that if we put $\sigma.v$ to be $\sigma.v = (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)})$, then we will actually get an action as defined in the lectures.

Problem 3. Prove that any group G acts on itself via $g.x := gxg^{-1}$.

The action of G on itself via $g.x := gxg^{-1}$ is called **conjugation**. Orbits of this action are called **conjugacy classes**.

Problem 4. Let G be any abelian group. Describe its conjugacy classes.

Prove that two permutations $s, t \in S_n$ belong to the same conjugacy class if and only if they have the same number of cycles of each length. Thus, the number of conjugacy classes in S_n is the number $p(n)$ of partitions of n.

Find conjugacy classes in S_3 and S_4.

Problem 5. Find conjugacy classes in the alternating group A_5. Are cycles (12345) and (12354) conjugate in A_5?

Problem 6. Let $GL_2(\mathbb{R})$ act on $\mathbb{R}^2$ in the usual way, i.e.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

Describe all orbits and stabilizers of this action. (Hint: there are two of them.)

Problem 7. Let $GL_2(\mathbb{Z})$ be the group of invertible 2×2 matrices with integer coefficients whose inverse also has integral coefficients. Let $GL_2(\mathbb{Z})$ act on $\mathbb{Z}^2$ the same way as $GL_2(\mathbb{R})$ acts on $\mathbb{R}^2$. Describe all orbits of this action and all stabilizers. (Hint: an orbit consists of vectors, whose coordinates have the same g.c.d.)